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Note on the undular jump

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A study is made of the asymptotic equations governing gravity waves on water which cause a transition from one surface level to a slightly higher one and approach a steady wave-form as time increases, at least at the head of the wave. A two-parameter family of limit processes is surveyed in each of which the time scale and the horizontal length scale tend to infinity in a definite relation to the amplitude, as that tends to zero. Small-amplitude linearization is shown to be possible at most during a transitory stage of the wave development. Arbitrarily close approach to steadiness at the head of the wave is found to imply that a substantial part of the transition wave must be ultimately governed by a near-steady variant of the non-linear equation of Korteweg & de Vries (1895) and must take the form of a train of cnoidal waves characterized by a parameter which changes slightly from crest to crest.

1. Introduction

In the theory of surface waves on water, certain basic wave forms play a fundamental role. One is the sinusoidal wave of small amplitude, another is the solitary wave. A third is the wave of transition from one water level to another, which is called a 'bore' or 'jump', if it raises the water level. It is known to show some analogy to gasdynamical shocks, but while their mechanism has been well explored, little is known about the internal structure of bores. The differences between them must be considerable, since experiment (Favre 1935; Ippen & Harleman 1956) shows the water surface to rise smoothly to a crest and then to settle down to the final level in a sequence of undulations, if the relative rise in level is sufficiently small.

An earlier theoretical study (Benjamin & Lighthill 1954) assumed that the waves are steady in the frame of a suitable Galilean observer, who sees a uniform cnoidal wave train determined by a small amount of energy dissipation at the very head of the wave, while dissipation elsewhere is disregarded. Such a theory, however, is unconvincing for lack of any physically founded mechanism from which the energy dissipation just at the front of the wave could be calculated. To put the following on a clear physical basis in this respect, only waves which propagate into fluid at rest will be considered. Since the water motion starts very gently at the front of such a wave, a boundary layer can be formed only quite gradually with increasing distance from the front, and viscous dissipation must be negligible near the front of the wave. It may become significant further away from the wave front,† but will be neglected below, since that is in any case proper for the first step in an analysis of the motion.

In fact, Benjamin (1962) subsequently modified the central notion of the earlier study in favour of conservation of energy, though not of total momentum. This leads to ideas rather similar to those based on radiation proposed by Lemoine (1948). But neither study presents more than the germ of a possible theory, which is not checked against the equations of motion. (Despite quite different premises, however, the present analysis will be seen to reproduce some of the ideas and analytical features of all the earlier work in modified form or interpretation. A similar, indirect influence of an unpublished report by Gardner & Morikawa (1960) should also be acknowledged.) Part of the real debt which I owe to Lemoine (1948) and Benjamin (1962) is the seed of doubt about the steadiness of transition waves. The present investigation therefore sets out to study the *asymptotic* behaviour of transition waves, as the time tends to infinity, and to ask whether any waves can, consistently with the full equations of motion, approach steadiness at least at the head of the wave? Here steadiness is of course understood to refer to the frame of a suitable Galilean observer, and the 'head' of the wave means, say, the part between the front and the first trough.

An analogy between bores and the more complicated transition waves in collisionless plasma has been uncovered by Gardner & Morikawa (1960), and Morton (1962) has made a thorough survey of the possible shock-discontinuities for that problem. The value of the survey was greatly enhanced by numerical computations of initial-value problems of the Korteweg–de Vries (1895) equation (which does not, however, apply initially to his plasma problem or its hydraulic analogue); it showed the solutions to approach asymptotically the predicted shocks—except for the smallest amplitudes, where numerical convergence was not achieved. In view of this additional indication that the key problem for an understanding of bore structure is posed by the weak bores, only transition waves of *small amplitude* are considered in the following.

The basic double limit, $t \to \infty$, $\epsilon \to 0$ (in this order!), where ϵ is the small-amplitude parameter, may be made more homogeneous by writing $t = T/\tau$ where τ is an artificial parameter, and letting $\tau \to 0$ for fixed T. The present investigation considers only simpler, single limits for which $\tau \to 0$ in some definite relation to ϵ, but approximates (Meyer 1966) the desired double limit by surveying a sufficiently large class of such relations $\tau(\epsilon)$. Indeed, the function class Γ admitted is that of all functions tending to zero continuously with ϵ and such that, for any pair $\theta(\epsilon)$, $\kappa(\epsilon)$ of members of the class, at least one of $\lim \theta/\kappa$ and $\lim \kappa/\theta$ exists. Different functions $(\tau(\epsilon))^{-1}$ then represent large times of different order of magnitude relative to ϵ^{-1}. Since initial conditions will not be considered, the meaning of the 'time scale' τ^{-1} will actually relate, not to time intervals, but rather to the largest rates of change occurring in the motion (for the precise definition, see §2); and 'larger time' is to be interpreted as decreasing unsteadiness.

† But Sturtevant (1965) has pointed out that the dominant effect of viscosity should cause an increase(!) in energy, mass flow and total momentum flow for the undulations following upon the first crest.

A complication arises from the fact that the head of the wave may be anticipated to be formed by the harmonic components of smallest wave-number. Much in contrast to the case of a gasdynamical shock, the discontinuity analysis (Rayleigh 1914) of weak bores cannot be linked to the idea that the transition takes place over a short distance. Rather, it amounts to a comparison of the initial and final equilibrium states based on the conjecture that the process is altogether steady, or at any rate, that mass and total momentum do not accumulate in the transition zone at a rate different from that corresponding precisely to the increase in width of this zone. In any case, long waves must be anticipated to form the head of the transition, and hence the relevant horizontal length scale there must be large compared with the vertical length scale. Their ratio may be characterized by a third parameter, δ^{-1}, which should be determined by the equations of motion and other data of the problem, and which cannot therefore represent a third, independent limit process. Thus if $\tau = \tau(\epsilon)$, then also $\delta = \delta(\epsilon)$. Since the functions of the class Γ can be naturally grouped in equivalence classes, which can be ordered (Kaplun & Lagerstrom 1957), the task before us amounts to a survey of a two-parameter family of single limit processes. Since the basic question concerns asymptotic wave-forms for large times, this task can be reduced somewhat by assuming $\tau(\epsilon)$ to have the form $\tau = \delta(\epsilon)\gamma(\epsilon)$, where $\gamma \in \Gamma$ is also a small parameter. Physically, this limits the survey to wave-forms for which the unsteadiness at the head of the wave has decreased below the level naturally associated with the small horizontal variations; and that would appear to be the proper interpretation of near-steadiness.

These notions will be formulated mathematically in § 2, and already the preliminary analysis of § 3 will show that δ^{-1} cannot be the only relevant horizontal scale. This is not quite unexpected. The conjecture that the mechanism of weak bores is *dispersive* has come to be generally accepted, due largely to the influence of Brooke Benjamin, and is strongly supported by Morton's (1962) and Peregrine's (1966) computations. It follows that the wave must be anticipated to assume a clear-cut form only asymptotically, as $t \to \infty$, and that the asymptotic description must be anticipated to be thoroughly non-uniform with respect to the horizontal co-ordinate, because roughly speaking, it should depend on the value of x/t. The following is concerned primarily with the head of the wave, and δ^{-1} is therefore to be understood as the smallest horizontal scale which is relevant *there*. It need not, and indeed turns out not to, be a wave-length.

The non-uniformity with respect to x precludes, in most of the present analysis, the use of the boundary condition specifying the final surface level achieved at the tail of the transition wave. It appears plausible, however, that the assumed near-steadiness of the head of the wave presupposes a similar near-steadiness of the final level, and for simplicity, wherever reference can be made to the tail boundary condition, it will be assumed to specify a surface level independent of time and exceeding the initial level H by an amount $O(\epsilon H)$.

Although the two-parameter family of limit processes represents a net cast very wide, it turns out to drag up only one asymptotic transition wave which (i) can raise the water level, (ii) has unsteadiness small compared with the horizontal variations at the head of the wave, and (iii) can be consistent with the equations of fluid motion. As expected, it will be found in §4 that the wave can be of Airy's type ('shallow-water' wave) for at most a brief period, if it raises the water level. It is shown in §5 that it cannot remain indefinitely of Jeffreys' type either (linear long wave with dispersion), however small the amplitude, because that type of transition wave spreads too fast. Ultimately, it must take the form of a wave-train, governed by a near-steady variant of the equation of Korteweg & de Vries (1895), in which each individual wave approaches a cnoidal wave characterized by a parameter F varying but slightly from one wave to the next (§6). The very first wave must approach the form of a solitary wave, and the separation of the first crests must grow large, at least like the logarithm of the ratio of the $-\frac{3}{2}$-power of the amplitude scale to the time scale.

These results show, incidentally, that the only possible near-steady transition wave starts with a wave-train of the kind envisaged by Whitham (1965), and the need for averaging over many waves follows directly from the result (§6) that appreciable changes in the cnoidal parameter F can occur only over such distances. However, the following shows that only one parameter is involved, not three, and our analysis yields automatically the equation (32), governing the growth of F, which must replace the approximation conjectures of Whitham (1965). That equation shows the unsteadiness of the transition—however small it may have become—to retain a substantial influence on the (inviscid) wave-train at a large distance from the head of the wave. Moreover, the local approach to steadiness is seen not to imply an approach to steadiness for any part of the wave-train comprising more than a few crests. In addition, the present analysis gives no indication that even the local approach to steadiness can be maintained uniformly throughout the transition wave. The complete wave is likely to be a puzzle in several pieces, and this paper only assembles the piece which must be laid down first, before others can be fitted in with confidence.

The results appear in qualitative agreement with what little is known experimentally. Indeed, they furnish an explanation for the marked difference between the wave-forms observed respectively by Favre (1935) and Ippen & Harleman (1956).

2. Formulation

Let x^* denote horizontal distance and y^*, vertical distance measured upward from the flat, horizontal bed supporting the fluid. For two-dimensional, incompressible motion, the corresponding velocity components are

$$u^* = \partial\psi^*/\partial y^*, \quad v^* = -\partial\psi^*/\partial x^*, \tag{1}$$

where $\psi^*(x^*, y^*, t^*)$ is the stream function and t^*, the time (stars will distinguish dimensional quantities). With viscosity neglected, a wave invading water at rest remains irrotational, so that

$$\partial u^*/\partial y^* = \partial v^*/\partial x^*, \tag{2}$$

and the equations of motion under the influence of gravity are

$$\partial u^*/\partial t^* + u^* \partial u^*/\partial x^* + v^* \partial u^*/\partial y^* = -\rho^{-1}\partial p^*/\partial x^*, \tag{3}$$

$$\partial v^*/\partial t^* + u^* \partial v^*/\partial x^* + v^* \partial v^*/\partial y^* = -\rho^{-1}\partial p^*/\partial y^* - g, \tag{4}$$

where p^*, ρ and g denote respectively the pressure and the constant density and gravitational acceleration. The bottom boundary condition may be written

$$\psi^*(x^*, 0, t^*) = 0, \tag{5}$$

and the exact boundary conditions at the free surface, $y^* = h^*(x^*, t^*)$, are

$$\partial h^*/\partial t^* + \partial\psi^*(x,^* h^*(x^*, t^*), t^*)/\partial x^* = 0, \tag{6}$$

$$p^*(x^*, h^*, t^*) = 0, \tag{7}$$

if surface tension and interaction between air and water are neglected. These equations are invariant to Galilean transformations with y^* fixed, and we choose an observer travelling in the direction of wave propagation with constant velocity

$$U = (gH)^{\frac{1}{2}}(1-\lambda)^{-\frac{1}{2}},$$

with respect to the water at rest (where the surface is at $y^* = H$), and require this velocity to be just sufficient for him to keep up with the head of the wave. It is natural to anticipate that λ may depend on ϵ and to assume $\lambda \in \Gamma$ (§1).

We introduce non-dimensional variables

$$x = \delta(\epsilon)x^*/H, \quad y = y^*/H, \quad t = \delta\gamma(\epsilon)Ut^*/H,$$

and a stretching transformation to non-dimensional deviations from the initial equilibrium,

$$\psi^*(x^*, y^*, t^*) = -Uy^* + \epsilon UH\psi(x, y, t; \epsilon),$$

$$\rho^{-1}p^*(x^*, y^*, t^*) = g(h^*-y^*) + U^2 p(x, y, t; \epsilon),$$

$$h^*(x^*, t^*) = Hh(x, t; \epsilon) = H(1+\beta(\epsilon)\eta(x, t; \epsilon)),$$

where also $\beta \in \Gamma$ (§1), if $\beta \to 0$ as $\epsilon \to 0$. It is convenient to write

$$\psi(x, h(x, t), t; \epsilon) = h(x, t; \epsilon)\,\overline{u}(x, t; \epsilon).$$

Equation (6) is then transformed into

$$(\gamma\beta/\epsilon)\,\partial\eta/\partial t + \partial q/\partial x = 0, \tag{8}$$

where $$q = \overline{u} - (\beta/\epsilon)\eta + \beta\overline{u}\eta$$

represents the scaled mass-flow imbalance, by comparison with a steady flow. The total head deviation is

$$\rho^{-1}p^* + \tfrac{1}{2}(u^{*2}+v^{*2}) + g(y^*-H) - \tfrac{1}{2}U^2 = \epsilon U^2 m,$$

in terms of which (3), (4) may, by (2), be written as

$$\partial u^*/\partial t^* + \epsilon U^2\,\partial m/\partial x^* = 0, \quad \partial v^*/\partial t^* + \epsilon U^2\,\partial m/\partial y^* = 0,$$

and these may be expressed in terms of the scaled velocity deviations, $u \equiv \partial\psi/\partial y$, $v \equiv -\partial\psi/\partial x$, by

$$\gamma\,\partial u/\partial t + \partial m/\partial x = 0, \tag{9}$$

$$\delta^2\gamma\,\partial v/\partial t + \partial m/\partial y = 0. \tag{10}$$

Finally, m is related to the scaled, non-dimensional observables by

$$m = (1-\lambda)(\beta/\epsilon)\eta - \overline{u} + r, \quad r = \epsilon^{-1}p + \overline{u} - u + \tfrac{1}{2}\epsilon(u^2+\delta^2v^2). \tag{11}$$

All these relations are exact; in particular, (8) to (10) express directly the principles of conservation of mass and momentum.

The limit processes sketched in §1 may now be formulated as follows. It will be assumed that (1) to (7) possess solutions for which ψ, p and η tend, as $\epsilon \to 0$, to functions of class C^∞ with respect to x, y, t for $0 \leqslant y \leqslant h$ on any compact set in the (x, t)-plane.† Moreover, the solutions are assumed non-trivial in the sense that $\overline{u}^2+\eta^2 \not\equiv 0$ and indeed, $|\overline{u}|$, $|\eta|$, $(\partial\overline{u}/\partial t)^2+(\partial\eta/\partial t)^2$ and $(\partial\overline{u}/\partial x)^2+(\partial\eta/\partial x)^2$ do not possess upper bounds tending to zero with ϵ on every compact x, t-set; this expresses the notion that ϵ and β represent the proper amplitude scales, and $(\gamma\delta)^{-1}$ and δ^{-1}, the smallest relevant time and horizontal length scales, at the head of the transition wave. And accordingly, the boundary condition ahead of the jump is assumed to be

$$\eta \to 0, \quad \overline{u} \to 0, \quad \partial\overline{u}/\partial x \to 0, \quad \partial^2\overline{u}/\partial x^2 \to 0 \quad \text{as} \quad x \to +\infty \quad \text{for all } t. \tag{12}$$

The aim of the analysis will be to identify the functions $\gamma(\epsilon)$, $\delta(\epsilon)$, if any, for which these assumptions can survive even cursory scrutiny, and to deduce the corresponding asymptotic forms of the governing equations (1) to (7).

3. Long-wave equations

The quantity r may now be estimated as follows. By (2), $\partial^2\psi/\partial y^2 = -\delta^2\partial^2\psi/\partial x^2$, whence

$$\left.\begin{aligned} \psi &= ys(x,t)-\tfrac{1}{6}\delta^2 y^3\partial^2 s/\partial x^2+O(\delta^4), \\ \overline{u} &= s-\tfrac{1}{6}\delta^2h^2\partial^2 s/\partial x^2+O(\delta^4), \\ \overline{u}-u &= \tfrac{1}{2}\delta^2(y^2-\tfrac{1}{3}h^2)\,\partial^2\overline{u}/\partial x^2+O(\delta^4). \end{aligned}\right\} \tag{13}$$

By (7), on the other hand,

$$r(x,h,t) = \tfrac{1}{3}\delta^2h^2\partial^2\overline{u}/\partial x^2+\tfrac{1}{2}\epsilon\overline{u}^2+\delta^2 O(\max(\delta^2,\epsilon)),$$

so by (10) and (11),

$$r(x,y,t) = \tfrac{1}{3}\delta^2h^2\partial^2\overline{u}/\partial x^2+\tfrac{1}{2}\epsilon\overline{u}^2+\delta^2 O(\max(\delta^2,\epsilon,\gamma)), \tag{14}$$

which is nearly independent of y, whence the same follows for m. Observe that (13), (14) convert (8) and (9) into (approximate) long-wave equations. Indeed, the limit process $\epsilon \to 0$ has been introduced only into the interpretation of u and r in terms of $\overline{u}$, which is obtained by vertical averaging of the equations of motion and use of the surface and bottom conditions, and the approximation is therefore valid on any compact x, t-set on which the limit process is justifiable—even if the set is so far from the head of the wave that the least value of $|x|$ in the set is not bounded. On any such set, therefore,

$$m+q = r-(\lambda\beta/\epsilon)\,\eta+\beta\overline{u}\eta = O(\max(\delta^2,\lambda\beta/\epsilon,\beta)), \tag{15}$$

† To avoid confusion, such sets must be kept distinct from others in which x or t grow beyond bounds as $\epsilon \to 0$. On the other hand, since the physical problem does not specify the x, t-origin, compactness can, in the first place, refer only to intervals. For definiteness, in the absence of explicit qualifications, compact sets are in the following understood to contain the x, t-origin, which is assumed to represent a point within the first hump or trough (whichever comes first) of the wave at a time when the degree of unsteadiness there is already small.

if that exists. This approach to long-wave equations, developed from a vertical averaging method communicated to the author by Dr Brooke Benjamin, is easily seen to be superior to most derivations of long-wave equations in the literature.

Some further properties may be established for the whole family of limits under investigation. Suppose $\lim \epsilon/\beta(\epsilon) = 0$; then from (8), $\partial\eta/\partial z_1 = O(\alpha_1)$ where $z_1 = \alpha_1 x$† and $\alpha_1^2 = \max(\gamma, \epsilon, \epsilon/\beta) \to 0$ as $\epsilon \to 0$. But by (12), $\eta \to 0$ while still $z_1 \to 0$, so that $\eta = O(\alpha_1)$ for all bounded z_1, contrary to the hypothesis that $|\eta|$ has no bound tending to zero with ϵ, and hence β/ϵ must be bounded. Again, suppose $\lim \epsilon/(\beta\lambda)$ exists, then from (8), (9), (11), $\partial\eta/\partial z_2 = O(\alpha_2)$, where $z_2 = \alpha_2 x$ and $\alpha_2^2 = (\epsilon/\beta\lambda)\max(\gamma, \delta^2, \epsilon, \beta) \to 0$ as $\epsilon \to 0$, whence again, by (12), $\eta = O(\alpha_2)$ for bounded z_2, contrary to hypothesis, and therefore $\lambda\beta/\epsilon \to 0$ and

$$\alpha \equiv \max(\beta, \delta^2, \epsilon, \lambda\beta/\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

From (8), (9) and (11), with $z = \gamma x$,

$$\left(\frac{1}{2}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)(m-q) = \frac{\partial}{\partial t}[\tfrac{1}{2}r + u - \bar{u} - \tfrac{1}{2}\beta\eta(\bar{u} - \lambda/\epsilon)] = O(\alpha),$$

so that $m - q = O(\alpha)$, for bounded z, because $m \to 0$, $q \to 0$ as $x \to +\infty$ for all t, by (11) and (12). By (15), therefore

$$m = \alpha M, \quad q = \alpha Q,$$

with bounded M, Q. But

$$M - Q = M + Q - 2\alpha^{-1}(\bar{u} - (\beta/\epsilon)\eta + \beta\bar{u}\eta),$$

whence $\bar{u} - \beta\eta/\epsilon = O(\alpha)$, and since an upper bound on $|\bar{u}|$ tending to zero with ϵ is excluded by hypothesis, ϵ/β must also be bounded. Thus β and ϵ belong to the same equivalence class in Γ,‡ and since only homogeneous boundary conditions have been formulated, no generality is lost by identifying them, whence

$$\beta = \epsilon, \quad \bar{u} - \eta = O(\alpha), \quad \alpha = \max(\delta^2, \epsilon, \lambda). \tag{16}$$

But then from (8), (9) and (11), $\lambda\partial\eta/\partial x = O(\max(\gamma, \delta^2, \epsilon))$, and since an arbitrarily small upper bound on $|\eta|$ is contrary to hypothesis, (12) implies that

$$\lambda/\max(\gamma, \delta^2, \epsilon)$$

is bounded as $\epsilon \to 0$.

For later reference, we note that (8), (9) and (11) imply

$$\left(\frac{1}{2}\frac{\partial}{\partial t} - \frac{1}{\gamma}\frac{\partial}{\partial x}\right)(\eta - \bar{u}) = \frac{1}{2\gamma}\frac{\partial}{\partial x}(r - \lambda\eta - \epsilon\bar{u}\eta) - \frac{1}{2}\frac{\partial}{\partial t}(\bar{u} - u), \tag{17}$$

if α/γ is bounded, whence from (8),

$$\left(\frac{\partial}{\partial t} - \frac{\lambda}{2\gamma}\frac{\partial}{\partial x}\right)\eta = \frac{1}{2}\frac{\partial}{\partial t}(\eta - \bar{u} + \bar{u} - u) - \frac{1}{2\gamma}\frac{\partial}{\partial x}(r + \epsilon\bar{u}\eta). \tag{18}$$

† $\partial/\partial z$ and $\partial/\partial z_i$ denote differentiation at fixed t, since z and $z_i (i = 1, 2, \ldots)$ are only scaled forms of x.

‡ The equivalence class of $\theta(\epsilon) \in \Gamma$ is defined as $\{\kappa(\epsilon) |$ both $\lim \kappa/\theta$ and $\lim \theta/\kappa$ exist$\}$.

To proceed further, it appears necessary to consider sub-families of our collection of limits. The largest values of γ for which ϵ, δ and γ all tend to zero are characterized by

$$\epsilon/\gamma \to 0, \quad \delta^2/\gamma \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{19}$$

Then α/γ is bounded, and the right-hand side of (17) remains bounded even after differentiation with respect to t. But by (12), $\partial(\eta - \overline{u})/\partial t \to 0$ as $x \to +\infty$ while $z = \gamma x \to 0$, so that $\partial(\eta - \overline{u})/\partial t = O(\gamma)$ for bounded x. The right-hand side of (18) then tends to zero with ϵ, and there are only two possibilities. If γ/λ is bounded, γ and λ belong to the same equivalence class in Γ, so that no generality is lost in taking $\lambda = \gamma$, and then from (18), the least upper bound of $|\eta|$, or briefly $\text{lub}\,|\eta| \to 0$, as $\epsilon \to 0$, contrary to hypothesis; the observer travels too fast and leaves the head of the wave behind. If $\lambda/\gamma \to 0$, on the other hand, (18) implies $\text{lub}\,|\partial\eta/\partial t| \to 0$, and since $\partial(\eta - \overline{u})/\partial t = O(\gamma)$, also $\text{lub}\,|\partial\overline{u}/\partial t| \to 0$, which contradicts the definition of γ; the observer cannot, consistently with the governing equations, expect to observe time-rates of change large enough to correspond to (19).

4. Airy waves

Consider next the subfamily of limits for which

$$\delta^2/\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{20}$$

Since $\epsilon/\gamma \to 0$ now implies (19), it may be excluded from further consideration, and no generality is lost in taking λ/ϵ to be bounded and $\alpha = \epsilon$. Then since $\overline{u} - \eta = O(\epsilon)$,

$$M + Q = \tfrac{3}{2}\overline{u}^2 - (\lambda/\epsilon)\,\overline{u} + O\,(\max\,(\epsilon, \delta^2/\epsilon)),$$

and from (8), (9),

$$\partial(M+Q)/\partial x = -(\gamma/\epsilon)\,\partial(\eta + u)/\partial t.$$

There are two possible cases.

(i) If $\alpha_4^2 \equiv \gamma/\epsilon \to 0$ as $\epsilon \to 0$ and $z_4 \equiv \alpha_4 x$, then $\partial(M+Q)/\partial z_4 = O(\alpha_4)$, and since (12) implies $M + Q \to 0$ and $\overline{u} \to 0$ as $x \to +\infty$ but $z_4 \to 0$, it follows that $\text{lub}\,|\overline{u}| \to 0$ as $\epsilon \to 0$, for bounded z_4, contrary to hypothesis.

(ii) If $\gamma = \epsilon$, (17) is valid and its right-hand side remains bounded even after differentiation with respect to t or x. Since $\partial(\eta - \overline{u})/\partial t \to 0$ and $\partial(\eta - \overline{u})/\partial x \to 0$ as $x \to +\infty$ but $z_5 \equiv \theta(\epsilon)x \to 0$, it follows that, if $\gamma/\theta \to 0$, both these derivatives are $O(\gamma/\theta)$ for bounded z_5 and hence

$$\begin{aligned}\partial(M+Q)/\partial x &= (3\overline{u} - \lambda/\epsilon)\,\partial\overline{u}/\partial x + O\,(\max\,(\delta^2/\epsilon, \gamma/\theta))\\ &= -2\,\partial\overline{u}/\partial t + O(\max\,(\delta^2, \gamma/\theta)).\end{aligned}$$

In the limit, therefore,

$$d\overline{u}/dt = 0 \quad \text{when} \quad dx/dt = (3\overline{u} - \lambda/\epsilon)/2, \tag{21}$$

and if our observer is to keep pace precisely with the head of the wave, where arbitrarily small values of $\overline{u}$ occur, his appropriate velocity is given by $\lambda = 0$, i.e. $U^2 = gH$. Moreover, if a secondary observer travels in the direction of x^* increasing with a velocity exceeding the local fluid velocity at his position by the local value of $(gh^*)^{\frac{1}{2}}$, then the primary observer sees him moving with velocity

$$dx/dt = \gamma^{-1}[\epsilon\overline{u} + (1-\lambda)^{\frac{1}{2}}(1+\epsilon\eta)^{\frac{1}{2}} - 1], \tag{22}$$

which takes the value specified in (21), in the case under consideration. Now, $(3\bar{u}-\lambda/\epsilon)$ is an increasing function of $\bar{u}$ and hence, if at some fixed t, $\bar{u}$ fails to be a monotone non-decreasing function of x for all bounded z_5—not to mention all bounded x—then a solution in the assumed class ceases to exist after a *finite* interval of t. The condition (20) is therefore consistent with the governing equations only if, at a sufficiently early, fixed time, $\bar{u}$ and η are monotone non-decreasing functions of x for all bounded z_5, and (21) then indicates that the same property is maintained for all later t.

Such waves which lower the surface level as they propagate into water at rest have long been known, and are relevant to the present investigation only to the extent that a wave ultimately raising the level might start with a phase lowering it temporarily. For this to remain consistent with the governing equations, no local raise of level interrupting the monotone property can occur at bounded t within bounded z_5-distance from the head of the wave. But then (21) shows the depressive part of the wave to spread so that $|\partial\eta/\partial t|$ and $|\partial\eta/\partial x|$ decrease as t increases, and any phase of the wave raising the level would not only have to lag far behind, but would also have to travel into water which, over any bounded x-interval, is ultimately in uniform motion. Such waves have been excluded (§1) by the assumption on which the neglect of viscous effects is based.

5. Jeffreys waves

For the subfamily of limits for which

$$\epsilon/\delta^2 \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{23}$$

the case $\delta^2/\gamma \to 0$ has already been dismissed in §3, and no generality is therefore lost in taking λ/δ^2 to be bounded and $\alpha = \delta^2$. Then

$$M+Q = \tfrac{1}{3}\,\partial^2\bar{u}/\partial x^2 - \delta^{-2}\lambda\bar{u} + O(\max(\lambda, \epsilon/\delta^2)),$$
$$\partial(M+Q)/\partial x = -\delta^{-2}\gamma\,\partial(\eta+u)/\partial t.$$

(i) If $\alpha_6^2 \equiv \gamma/\delta^2 \to 0$ as $\epsilon \to 0$, then since $M+Q \to 0$ as $x \to +\infty$,

$$\tfrac{1}{3}\,\partial^2\bar{u}/\partial x^2 - \delta^{-2}\lambda\bar{u} = O(\max(\lambda, \epsilon/\delta^2, \alpha_6)$$

for bounded $z_6 = \alpha_6 x$, and $\bar{u} \to 0$ as $x \to +\infty$. Thus if $\lambda/\delta^2 \to 0$, we may again deduce from (12) that lub $|\bar{u}| \to 0$ with ϵ, contrary to hypothesis. But if $\lambda = \delta^2$, then $\bar{u}\exp(+x\sqrt{3}) = $ const. and $\bar{u}$ does not remain bounded as $x \to -\infty$. Therefore, ϵ cannot be the proper amplitude scale of the bore, if the transition is monotone, or of the first crest, if it is not monotone. The result may, however, represent the outer skirt of a wave, at its front, and is indeed the first asymptotic approximation to (30) as $x \to +\infty$.

(ii) If $\gamma = \delta^2$, we may again appeal to (17), and an argument analogous to that of §4 shows that

$$2\,\partial\bar{u}/\partial t - \delta^{-2}\lambda\,\partial\bar{u}/\partial x + \tfrac{1}{3}\,\partial^3\bar{u}/\partial x^3 = O(\max(\gamma/\theta, \delta^2, \epsilon/\delta^2))$$

for bounded $z_5 = \theta(\epsilon)x$, if $\gamma/\theta \to 0$. The appropriate speed for an observer not outpacing the head of the wave corresponds to $\lambda/\delta^2 \to 0$, since even the longest waves of such small amplitude are found to propagate with a velocity differing ultimately by $o(\delta^2)$ from $(gH)^{\frac{1}{2}}$.

The limiting equation,

$$6\,\partial\overline{u}/\partial t+\partial^3\overline{u}/\partial x^3 = 0, \tag{24}$$

was studied (implicitly) by Jeffreys (1946), and integration of his result yields the basic solution (Gardner & Morikawa 1960 and Benjamin & Barnard 1964)

$$\overline{u}(x,t) = c\int_\sigma^\infty \mathrm{Ai}\,(\tau)\,d\tau, \quad \sigma = (2/t)^{\frac{1}{3}}\,x, \tag{25}$$

of (24), representing a wave of transition from the surface level H to the level $H(1+\epsilon c)$; Ai here denotes the Airy function (Jeffreys & Jeffreys 1946). This is a case, then, in which the tail boundary condition can be directly relevant to an analysis of the motion at distance $O(\delta^{-1}H)$ from the head of the wave, and it will be assumed to be

$$\eta(x,t)\to\eta_f = \text{const.} > 0 \quad \text{as} \quad x\to-\infty \quad \text{for all } t.$$

Since (24), (12) and this boundary condition are invariant under the transformation $x = a^{\frac{1}{3}}\xi$, $t = a\tau$, $\overline{u}(x,t) = v(\xi,\tau)$ for arbitrary constant a, the solution $\overline{u}(x,t)$ for the initial condition $\overline{u}(x,0)$ may be obtained by this transformation from the solution $v(\xi,\tau)$ for the initial condition $v(\xi,0) = \overline{u}(a^{\frac{1}{3}}\xi,0)$. As $a^{-1}\to 0$, $v(\xi,0)$ tends to the step function $\eta_f H(-\xi)$, and that is the initial condition satisfied by (25) if $c = \eta_f$. Any transition wave governed by (24) may therefore be anticipated to approximate (25) for large t.

It is important to note, however, that (24) can describe only a transitory stage in the development of a transition wave, not a true asymptotic development (Ursell 1953). In the case (ii) under discussion, the time scale, $(\gamma\delta)^{-1}$, equals the cube of the horizontal length scale, δ^{-1}, and thus (23) requires the time scale to be small compared with $\epsilon^{-\frac{3}{2}}$, where ϵ is the amplitude scale. Since $\overline{u}$ and η depend only on $t^{-\frac{1}{3}}x$, at least asymptotically, the wave preserves the relation $\gamma = \delta^2$, but $|\partial\overline{u}/\partial t|$ and $|\partial\overline{u}/\partial x|$ decrease as t increases, while the amplitude—represented by $\overline{u}(-\infty,t) = \eta_f$—is preserved, so that (23) must ultimately give way to $\gamma = \delta^2 = \epsilon$. (The same conclusion actually follows for any transition wave once characterized by (23) and of fixed amplitude and increasing time scales, from the earlier result that the alternative developments, $\epsilon/\delta^2\to 0$ with $\gamma/\delta^2\to 0$ or $\delta^2/\gamma\to 0$, do not admit transition waves.)

This conclusion is of disturbing relevance to the classical theory of small-amplitude dispersive waves, the literature on which is predominantly concerned with asymptotic solutions as $t\to\infty$. That theory therefore considers the double limit $\epsilon\to 0, t\to\infty$, in this order, whereas the reverse order is the physically relevant one. Our results show the order of the limits to be definitely non-interchangeable for transition waves on water; for sufficiently small amplitude, the effects of dispersion may be dominant for a long time, but ultimately, the cumulative effect of the small non-linearity must become important. It is natural that this may have a striking effect on transition waves, for which the amplitude scale does not change with time. Ursell (1953) has pointed out that it is relevant also to some aspects of waves the amplitude of which decays in time.

The results of this section may also help to explain the apparent discrepancy between the experimental results of Favre (1935) and of Ippen & Harleman

(1956). The latter studied an oblique hydraulic jump, steady with respect to a stationary observer, on a running stream and found a wave form qualitatively resembling (25). For such an experiment, distance along the wave front corresponds to our t, and that distance is limited by the width of the channel. Favre's experiment, on the other hand, concerns a bore travelling into water at rest in a very long channel. His findings bear little resemblance to (25), but appear in qualitative agreement with the results of the next section.

6. K-V waves

There remains only the subfamily of limits for which

$$\epsilon = \delta^2, \tag{26}$$

and since the case $\epsilon/\gamma \to 0$ has been dismissed in § 3, we may take λ/ϵ to be bounded, and $\alpha = \epsilon$. Thus

$$M+Q = \tfrac{1}{3}\,\partial^2\overline{u}/\partial x^2 + \tfrac{3}{2}\overline{u}^2 - \lambda\overline{u}/\epsilon + O(\epsilon),$$

$$\partial(M+Q)/\partial x = -(\gamma/\epsilon)\,\partial(\eta+u)/\partial t. \tag{27}$$

(i) If also $\gamma = \epsilon$, (17) is valid and an argument analogous to that of § 4 leads to

$$\frac{\partial\overline{u}}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{3}\frac{\partial^2\overline{u}}{\partial x^2} + \tfrac{3}{2}\overline{u}^2 - \frac{\lambda}{\epsilon}\overline{u}\right] = O(\epsilon/\theta), \tag{28}$$

the equation of Korteweg & de Vries (1895), for bounded $z_5 = \theta x$, if $\epsilon/\theta(\epsilon) \to 0$ as $\epsilon \to 0$. Apart from waves of uniform translation the only known, exact solution (Gardner & Morikawa 1960) has amplitude decreasing indefinitely with time. A numerical treatment by Morton (1962) was successful, except in the case here studied. The recent study by Peregrine (1966) traces the development of an Airy wave (§ 4) through one of Jeffrey's type (§ 5) to one beginning to resemble case (ii) below at the time at which the computation was broken off.

(ii) If $\gamma/\epsilon \to 0$, so that the time scale is large compared even with $\epsilon^{-\frac{3}{2}}$, then $M+Q = O(\kappa)$ for bounded $z_7 = \theta x$, if $\theta\kappa = \gamma/\epsilon$ and $\theta(\epsilon) \to 0$, $\kappa(\epsilon) \to 0$, as $\epsilon \to 0$. Thus

$$\tfrac{1}{3}\,\partial^2\overline{u}/\partial x^2 + \tfrac{3}{2}\overline{u}^2 - \lambda\overline{u}/\epsilon = O(\max(\epsilon, \kappa)), \tag{29}$$

which is Boussinesq's equation of solitary-wave theory. By (12), the only possible waves approaching steadiness so closely that $\gamma/\epsilon \to 0$ as $\epsilon \to 0$, must therefore approach solitary waves

$$\overline{u}(x) = k\,\mathrm{sech}^2((3k)^{\frac{1}{2}}x/2), \tag{30}$$

of amplitude parameter $\qquad k = \lambda/\epsilon,$

for x bounded below. But this conclusion does not follow for large negative x, and indeed—except for large positive x at the front of the wave—any interpretation of it must take account of the delicacy of (29) stressed by Benjamin & Lighthill (1954).

We recall therefore that (29) really stands, by (14), (16), (26), and (27) for the *near-steady* form

$$\tfrac{1}{3}\,\partial^2\overline{u}/\partial x^2 + \tfrac{3}{2}\overline{u}^2 - k\overline{u} = F, \tag{31}$$

$$F = M+Q+(k-\overline{u})(\eta-\overline{u}) + \tfrac{1}{3}\,\partial^2\overline{u}/\partial x^2 + \tfrac{1}{2}\overline{u}^2 - r/\epsilon,$$

$$\frac{\partial F}{\partial x} = -\frac{\gamma}{\epsilon}\frac{\partial}{\partial t}(\eta+u) + (k-\overline{u})\frac{\partial}{\partial x}(\eta-\overline{u}) + O(\epsilon), \tag{32}$$

of the equation of Korteweg & de Vries. From (30), existence of a non-trivial asymptotic solution corresponding to $\gamma/\epsilon \to 0$ requires $k > 0$. By (12), integration of (31) with respect to x gives

$$(\partial u/\partial x')^2 = u^2 - u^3 + 2fu - 2y \equiv C_x(u), \tag{33}$$

where $$u = \overline{u}/k, \quad x' = x(3k)^{\frac{1}{2}}, \quad f = F/k^2, \quad y = -\int_{x'}^{\infty} \frac{\partial f}{\partial x} u \, dx,$$

and (32) shows $C_x(u)$ to be a cubic in u with coefficients that tend asymptotically to constants over any finite interval of x. The criterion for two distinct real roots is

$$\Delta = (1 + 9f - 27y)^2 - (1 + 6f)^3 \leqslant 0, \tag{34}$$

and the curve $\Delta = 0$ in the (f, y)-plane is shown in figure 1 (which is of course related to the diagram of Benjamin & Lighthill (1954)). By (12), the front of the

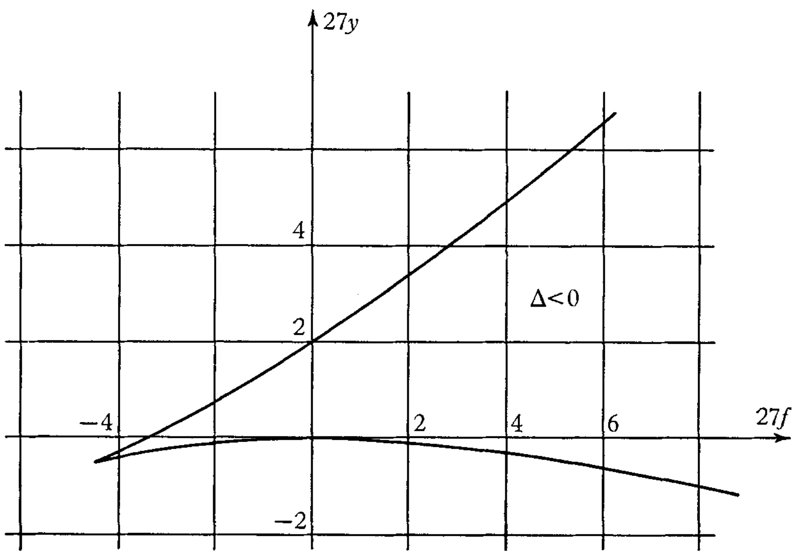

FIGURE 1

wave is represented by the origin of the (f, y)-plane, which corresponds to the solution (30) of (33). As x decreases, the representative point (f, y) must be anticipated to shift slightly from the origin, according to (32). If it moves into the region $\Delta < 0$, then (33) is the equation of a cnoidal wave

$$u = \rho + (1 + \nu - \rho)\,\mathrm{cn}^2(\tfrac{1}{2}x(1 + \nu - \sigma)^{\frac{1}{2}}), \tag{35}$$

of modulus $(1 + \nu - \rho)^{\frac{1}{2}}/(1 + \nu - \sigma)^{\frac{1}{2}}$, where $1 + \nu$, ρ and σ denote the roots of the cubic $C_x(u)$, in decreasing order. If ϵ is sufficiently small, then f and y (and hence also ν, ρ and σ) are of arbitrarily small magnitude, and the wave length

$$L \sim -\log\left(\frac{\rho - \sigma}{16}\right)^2 \sim \log\left[(y + \nu^2/8)/32\right]$$

is large, and (35) approaches (30).

On the other hand, if the representative point moves into the region $\Delta > 0$, the solution of (33) and (12) fails to be bounded for all bounded x and hence, no solution of the governing equations consistent with $\gamma/\epsilon \to 0$ exists. Conversely, since $\Delta(f,y) = 0$ implies $3\,dy/df = 1-(1+6f)^{\frac{1}{2}}$ and $d^2y/df^2 = -(1+6f)^{-\frac{1}{2}}$, while $\partial y/\partial x = u\,\partial f/\partial x$ and $u \to 0$ as $x \to +\infty$, the path of the representative point is tangent to the curve $\Delta = 0$ at the origin, and a necessary condition for a solution consistent with $\gamma/\epsilon \to 0$ is therefore that $1+k(\partial\overline{u}/\partial x)/(\partial F/\partial x)$ have a non-negative lower bound as $x \to +\infty$.

In conclusion, the investigation has contributed to a clarification of the issues by qualitative identification of the front part of the only asymptotically steady jump possibly obtained from our two-parameter family of single limits. Since the results appear compatible with the experimental evidence so far available, the next task would appear to be a study of (32), and until this reaches a stage where direct use can be made of the tail boundary condition, the amplitude k may remain undetermined. The traditional value is $k = 3\eta_f/2$ (Rayleigh 1914).

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